The shape of the universe

Paul Wesson shows how new techniques enable us to obtain accurate pictures of the Big Bang.

Abstract

Diagrams that accurately show essential physical features of models can be a valuable aid to understanding. Seeing the consequences of changes to the models is a powerful way to understand relationships that are otherwise evident only from complex calculations. Here I present a new way to visualize the shape of the universe, in a mathematically accurate way, using plots that incorporate the relationships between the fourth and fifth dimensions deduced from membrane theory and induced-matter theory. Among other things, these plots show a new picture of the Big Bang.

What is the shape of a soccer ball? Spherical, you correctly reply. But there are subtleties of perception in your answer: what you see is a surface at a constant distance from a centre, so you have embedded a 2-D space of constant curvature in a 3-D space which is flat.

Okay. Now think about the 3-D space of everyday perception embedded in the 4-D spacetime of Einstein's theory of general relativity. This is a bit trickier. Ordinary space does not have strange properties near to us, which is equivalent to saying that it is close to flat. But over cosmological distances it may be curved. The shape of the 3-D part of the 4-D manifold can be closed (like a ball), flat (like a table) or open (like a saddle). Even if 3-D space is flat, 4-D spacetime is not: matter implies curvature, so the universe has a definite shape.

Let us now go to five dimensions. Here things start to get a little difficult to visualize. However, recall the soccer ball: we intuitively recognize it for what it is because our minds automatically embed it in a flat space with one extra dimension (2-D in 3-D). Another way to realize that the surface of the ball is curved is more technical, but analogous to what astronomers do when they observe objects such as quasars in the universe. Imagine you are an ant wandering about on the surface of the soccer ball. You never reach an edge (the universe has no edge). If you trace out a triangle on the surface, the sum of its angles is not 360° (the universe is not flat). This technical way of deciding the shape of something is valid, but not intuitive. Even for professionals, it would be helpful if there were some way to embed the universe in a flat background and simply see its shape.

Recently we figured out how to do this in a mathematically accurate manner, and among other things obtained pictures of the Big Bang.

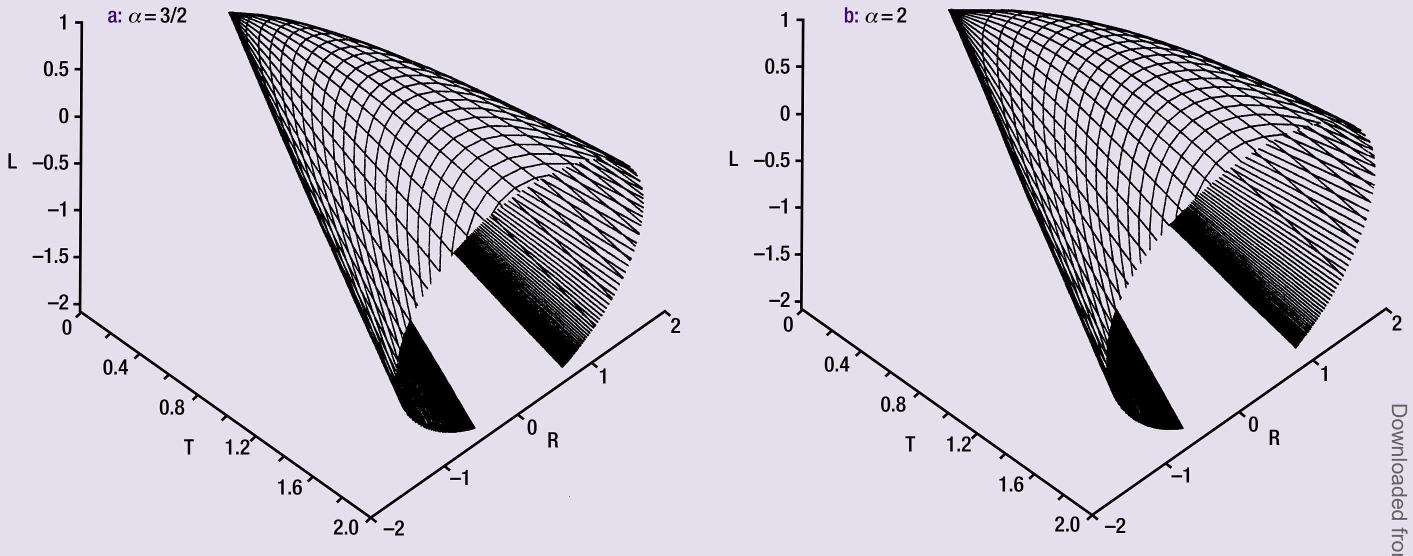
History and *N* dimensions

History involves us imaginatively travelling on the axis of time (t), with subsidiary information about space (x,y,z). Thus in 1492 (measured from an arbitrary datum we refer to as AD), Columbus sailed the ocean blue (going from Europe to America). And a photon from a quasar typically left it about 10 000 million years ago, arriving in a telescope on Earth after a journey of 10 000 million light years. Clearly, time and space are part of the same construct. Minkowski realized this, when he multiplied the speed of light c (about 300 000 km s^{-1}) on to the time, forming ct, which he used as a coordinate on the same footing as the length, breadth and depth we label as x,y,z. This is the basis of special relativity, which we associate primarily with Einstein, though previous work had been done by Lorentz, Fitzgerald and Poincaré. This theory describes photons (zero mass) moving in flat ordinary space (no matter). It was, however, the genius of Einstein alone who realized that this geometrical approach needed to be taken further, to include massive particles moving through space filled with matter. This requires that the manifold obtained by joining time (1-D) and space (3-D) has to be curved.

General relativity is correct, as far as it goes. It has been extensively tested in the solar system, using the perihelion advance of Mercury, the redshift of light from the surface of the Sun, and the deflection and time-delay of photons passing near the Sun from remote astronomical objects. It has recently been tested further by cosmological data. The standard cosmological models, based on Einstein's theory, were developed by Friedmann, Robertson, Walker and Lemaitre. The galaxies in these models are distributed like a fluid with no centre and no edge (we have never observed either), and the propagation of light from very remote sources shows bending and lensing effects in agreement with the theory. However, Einstein's theory is mathematically complicated, involving in the general case 10, second-order, nonlinear, partial differential equations. Also, it is hard for us two-eyed creatures to visualize curved 4-D spacetime. It would be helpful in the latter regard to have something like the image of the soccer ball we noted above.

The required extension of geometry from 4-D to 5-D was made by Kaluza in 1921. At least, that was when his paper was published. History has it that Einstein "sat on it" for a while as referee. (Einstein actually liked the idea of extending spacetime to higher dimensions, and worked on it in his later life.) Kaluza was motivated by the wish to unify gravitation as described by Einstein's equations with electromagnetism as described by Maxwell's equations. This works well for the fields but has problems with their sources, particles. These problems were apparently solved by Klein in 1926. He proposed that the extra dimension was rolled up, or "compactified", to an unobservably small size. This traditional Kaluza–Klein theory has an enormous literature. But the theory is not correct. In recent times it has been realized that it leads to wrong values for the masses of elementary particles (the hierarchy problem) and the wrong value for another parameter which enters the theory (the cosmological constant problem). Attempts to solve these problems have been made, chiefly along the lines of still further extending the dimensionality of the theory. Hence 10-D superstrings, 11-D supergravity and other geometry-based theories aimed at unifying gravitation, electromagnetism and the strong and weak interactions of particle physics.

But, hold it! If the world is 5-D (say), and we do not "see" the extra dimension, does



1: **Plots showing the shape of the universe for a variety of values of the constant α for various values of the other parameter (l_0 = 20, 40, 60).**

this necessarily mean that the latter has to be compactified or tiny?

No. Recall the ant on the surface of the soccer ball. It does not care about the distance to the centre, because it is constrained to be on the surface.

Aha! Enter "induced-matter" theory in 1992 and – more recently – "membrane" theory. These two theories are actually similar, in concept and formalism. In the former, what we call matter in 4-D is a consequence of (or induced by) the fifth dimension; while in the latter, matter is on a 4-D membrane in a 5- (or higher) dimensional world. Distances in induced-matter theory are calculated most readily using the so-called canonical metric, while distances in membrane theory are calculated using the so-called warp metric, the two being related mathematically. Kind critics might refer to this dualism as a case of great minds thinking alike; while less-kind critics might refer to both schools as having taken 70 years to begin to see the wood for the trees. Whatever the view, physicists are now in the happy position of having a way of dealing with 4-D physics in a 5-D way which is mathematically consistent, and (because Einstein's theory is embedded without violence) observationally acceptable.

The main difference between induced-matter theory and membrane theory is how they handle energy. This might be the common-or-garden kind of energy present in the rest masses and motions of particles, or exotic forms of energy associated with what older generations called "vacuum" (which we now know is not really empty). In induced-matter theory, the starting equations look like ones for 5-D empty space, but these break down naturally into 4-D ones which include those of Einstein and mimic exactly the known properties of energy. This realizes Einstein's dream of matter from geometry. The mathematical basis for this actually goes back to an erudite person called Campbell, who in a book published in 1926 proved that it was always possible to go from an empty theory in *N*-D to one with matter in (*N*–1)-D. Nowadays, we can derive Campbell's theorem more succinctly using results on embeddings due to Arnowitt, Deser and Misner. (But hats off to Campbell anyway!) By contrast, membrane theory takes equations in 5-D or *N*(>5)-D, and puts the matter in explicitly, or by hand so to speak. In this theory, the matter is constrained to lie on the membrane by forces. As stated, the two approaches are similar mathematically, but clearly different conceptually.

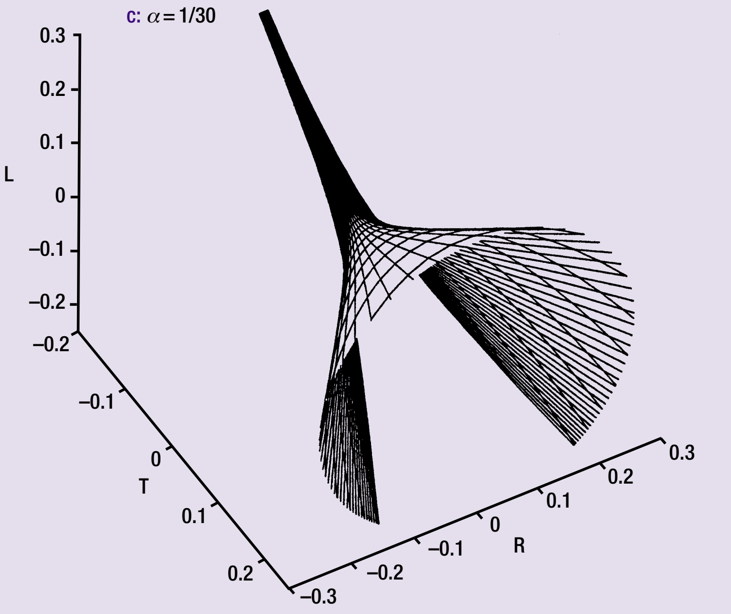
Whichever way we view it, the fifth dimension in these new theories is in general curved. Putting a curved 4-D universe in a curved 5-D manifold may be an interesting exercise in differential geometry, but lacks a certain pizzazz. However, we can get back to our alluring analogy with the soccer ball by using some results from modern cosmology.

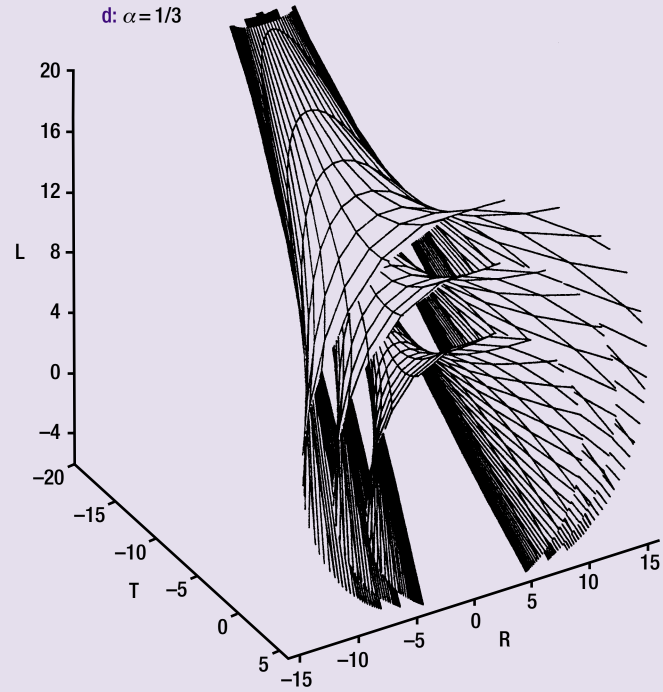
Data and maps

The real universe, by design or accident, is very close to uniform. For example, the galaxies are clustered; but over the largest distances we can probe with telescopes, their density has negligible fluctuations. Also, the radiation of the 3 K microwave background, which fills the cosmos, has fluctuations in temperature of only a few parts in 100 000. From data on both galaxies and radiation, we infer that in addition to what we see with telescopes there are large amounts of dark matter. We do not yet know exactly what this is made of, but from its gravitational effects we infer that it too is distributed quite uniformly. Modern data from the lensing of remote galaxies and QSOs also indicate a significant contribution from the energy density of the "vacuum", which in Einstein's theory is measured by the cosmological constant. Importantly, the sum of all of these kinds of energy makes the total density very close to a special or critical value.

This means that, according to general relativity, the 3-D (ordinary) space part of 4-D spacetime is flat. That is, there is a flat 3-D space embedded in a curved 4-D manifold. These considerations of uniformity and spatial flatness turn out to be very lucky in our quest to see the shape of the universe: the strong symmetries of the actual universe mean we can in principle embed it in a 5-D manifold that is flat.

So how exactly do we do this? The answer is that we need a map.To see that the problem is non-trivial, consider the Earth (or a soccer ball). The Mercator projection, which is used in school atlases, is useful because it maps the curved surface of our globe on to a flat page. However, in so doing it distorts the areas of the land masses, making them larger at higher latitudes than near the equator. (It is said that the British liked this mapping because it exaggerated the size of their empire, but while it helped with Australia they never owned the apparently huge Greenland.) This kind of mapping problem can be solved with an Equal-Area projection which, while it may look odd, is used by geographers. There are indeed an infinite number of ways of making a map, either for the Earth or the universe. The value of a map





depends on what it will be used for.

To see the shape of the universe, we have a clear plan: there is a flat 3-D space that is part of a curved 4-D spacetime and we wish to embed the latter in a flat 5-D manifold.

The equations needed to do this are, unfortunately, not as simple as the concept. In general relativity, there are several results known about embeddings (which is a posh term for maps). Thus the Schwarzschild solution, which describes the gravitational field outside the Sun, was shown by Tangherlini in 1963 to be embeddable in a flat higher-dimensional manifold with $N \geq 6$. And *any* solution of 4-D Einstein theory can be embedded in a flat manifold with $N \geq 10$. (This is one of the main motivations for superstrings, where so-called zero-point energies are cancelled out in a manifold with $N = 10$.) Our problem is topologically a bit simpler, because the symmetries of the actual universe outlined above mean we only have to consider a flat 5-D manifold. However, because we are trying to represent the real universe and not some hypothetical construct, we have to ensure that we do not contradict the physics involved. The physics is encapsulated in field equations. In 4-D general relativity, these are the 10 Einstein equations alluded to before. In an N-D theory of the Einstein type, the number of field equations is $N(N-1)/2$. This gives 15 for $N = 5$ as in induced-matter or membrane theory. So before we rush off into physical 5-D oblivion, we need a solution of the field equations that represents 4-D reality. Relax. The solution was found by Ponce de Leon in 1988 and shown to agree with cosmological data by Wesson in 1992. It has many interesting properties that go beyond standard cosmology, but for the present purpose all we need to know is that the solution guarantees reality.

Now we can sharpen our pencils! Safe with the physics, we can apply a bit of IQ and construct the map.

The shape of the universe

A soccer ball on a table is easy to visualize, but we should not expect the universe to be so simple. At the present epoch, the energy in the 3 K microwave background is many orders less than that in other forms of matter, a situation that can be summed up by saying that the equation of state (a relation between pressure and density) is analogous to that of dust.

At early epochs, the energy in radiation and the motions of ultrarelativistic particles was comparable to that in the rest masses of the particles, and the equation of state was close to that of photons, where the pressure is 1/3 of the energy density.

At extremely early times, we are not sure of the equation of state. But data from the microwave background indicate that there may have been a period of very rapid expansion, called inflation. This can be handled using a classical description of a fluid if the pressure is allowed to be negative. This may sound odd, but just means that the particles in the Big Bang fireball were trying to pull each other together, even though the universe was outpowering this by its expansion. (The cosmological constant, referred to before, can in Einstein's theory be recast as describing a kind of vacuum fluid where the pressure is proportional to *minus* the energy density.) This is an uncertain section of cosmology, but an exciting one where classical physics meets particle physics; so if we wish to cover all the bases, we should think about the shape of the universe at these extremely early times.

What the above means is that the shape of the universe evolves with time. There are at least three phases. These depend on the equation of state, and in the Ponce de Leon models are specified by three values of a constant alpha: $\alpha = 3/2$ for dust, $\alpha = 2$ for radiation, and $\alpha =$ small (but unknown) for inflation. This step-like description is, of course, an approximation to what really happened; and if we had better astrophysical data we could smooth the picture. It would look like one of the blobs in a 1960s lava lamp.

This analogy is closer than most. Consider a blob of oily material floating in water. The surface of the blob changes its shape depending largely on the physics at the surface, while the water stays there. Substitute the surface of the blob for 4-D spacetime, and the water for the flat 5-D manifold. Voila!

In general relativity and its N-dimensional extensions, things happening in one less dimension are said to occur on a hypersurface. Therefore, in induced-matter and membrane theory in 5-D, our universe in 4-D is a hypersurface. The map we are interested in is one where there are coordinates for a flat 5-D manifold, and other coordinates for a curved 4-D hypersurface which we call spacetime. We could use Cartesian coordinates to describe things; but as in cases like the Earth or a soccer ball, we can make life easier by using spherical ones to describe ordinary space. And because the real universe is uniform (see above) we only really need one of these, the radius. Let us label the essential coordinates in the big 5-D space as the time T, radius R, and an extra length L. The corresponding ones in the little 4-D space are the time t and the radius r. (The number of coordinates in the big space must be one larger than the number in the little space, because we are "going on to" a hypersurface. This is like emerging from a trapdoor on to a curved deck.) In the big space, nothing much happens because T, R, L are labels for a static, flat manifold. In the little space, galaxies evolve in time t and are located at different distances r in ordinary space. Mathematically, the view from 5-D to 4-D is encoded in functions which map T, R, L to t, r, and these functions are complicated.

So let us skip them and instead use pictures that contain the same information. Before showing these, however, a few short comments are in order about just how the functions turn into pictures. In the 3-D world we use t and r as coordinates because they are the most convenient. (We can then ignore the angular spatial measures because – again using the soccer-ball analogy – they make no difference for an ant on the surface.) In the 5-D world, we could if we wished use Cartesian coordinates in addition to the time, because the big space is flat. However, this is not convenient because it makes unnecessarily complicated the correspondence between the big and little spaces. Hence we match t, r to T, R, L. The last is hard for us to visualize, but is nevertheless just an ordinary length. The correspondence between the coordinates in the 4-D and 5-D spaces, as constrained by the underlying physics, then gives us pictures which are analogous to the Penrose diagrams found in books on general relativity (these help us visualize 4-D physics on a 2-D sheet of paper). However, while the latter can often be drawn by hand, our functions are so complicated that they have to be drawn by computer and, indeed, even modern computers cannot fill in every cranny of the plots. With these comments understood, let us take a look at some of the pictures in our gallery.

Figure 1a shows the shape of the universe at the present epoch ($\alpha = 3/2$, dust). The 5-D axes are labelled T, R, L as described above. The roughly cone-shaped structure is our 4-D universe. The lines along its length and around its circumference show how the galaxies evolve and where they are located. The shape (in cross-section) is a parabola. Where everything starts in 4-D is the Big Bang.

Figure 1b shows the shape of the universe at early epochs when it was extremely hot ($\alpha = 2$, radiation). The labelling is as previously, but the shape is different.

Figure 1c shows an extremely inflationary universe ($\alpha = 1/30$). The distinctive trumpet shape is due to the effects of interactions between particles involving the vacuum, whereby matter is pushed apart much more vigourously than in other models. The Big Bang, in so far as it can be defined, is now at "past null infinity" or indefinitely ancient.

Figure 1d shows a "nest" of moderately inflationary models ($\alpha = 1/3$, with the hypersurface located at values of 20, 40 and 60 times the unit value used in previous figures). The match between classical cosmology as described by general relativity, and quantum cosmology as described by N-dimensional field theory, is still being worked on, and we do not know exactly what shape the universe had in extreme history.

The preceding four figures, even if they have no other merit, are a big advance in visualizing the universe. By embedding it in (or mapping it from) 5-D, we can actually see its 4-D shape. But we may not be talking here only about a pretty face: what if the fifth dimension is real?

Summary and speculation

If you look at a soccer ball, it is easy to determine its shape because you see it in a flat 3-D background. It is not easy to visualize the shape of the 4-D universe, because it is curved and we are *in* it. However, the nature of the real universe as inferred from observations allows us to embed it in a flat 5-D manifold. The shape so revealed is not as simple as that of a soccer ball, and evolves with time; but still, pictures like those given above are informative.

Modern cosmology makes good use of extra dimensions. 5-D Kaluza–Klein theory is the basic extension of 4-D Einstein theory. Its two current forms are induced-matter theory (in which energy and matter are geometrical in origin) and membrane theory (in which matter is on a brane or hypersurface). 10-D and 11-D take things further, as in superstrings and supergravity.

But there is a fundamental question involved which is not so much technical as conceptual: are extra dimensions merely hypothetical constructs, or are they (in some sense) real?

It is educational before risking an opinion on this to inquire what "real" means in physics and astronomy. Is an electron "real", as envisaged in Bohr's model of the atom as a miniature solar system; or is it a shadowy thing, as described by the orbital wave functions of Heisenberg and Schrodinger? Is a planet like Pluto "real" because we infer its existence from perturbations in the orbits of other objects, in a solar system whose gravitational laws are established; or do we only admit its existence after we have seen it via light in a telescope?

Most physicists and astronomers today would say that something is "real" if we infer something new that has a logical explanation and does not wreck what we already accept. The neutrino is a good example of this philosophy: it was postulated to exist because of our belief in the principle of the conservation of energy, and later confirmed by experiments with new kinds of detectors. This philosophy has gradually become accepted over the last 70 years, and can be traced back to the work of Einstein and his contemporary Eddington. Both men realized that there is more to physics than the dreary mechanics of the ether, and opened the way to new and more imaginative ways of describing the physical aspects of our existence. If we acknowledge that physics involves not only things, but also imagination, what opinion should we have about extra dimensions?

In 5-D induced-matter theory, the fifth dimension is all around us: it is the energy of the world, whether in the rest masses of particles, the kinetic energy of their velocities, the potential energy of their interactions, or extra contributions involving what has traditionally been called the vacuum. In 5-D membrane theory, or higher-dimensional versions of it, the extra parts of the manifold are not apparent to the eye, but control the interactions of particles and therefore ultimately the matter of everyday existence.

If higher dimensions are merely inventions, they are uncannily clever ones. History, common-sense and data indicate that they may well be "real".

If the fifth dimension is real, the implications are profound. Intensive work is underway on the technical implications of $N(>4)$-D physics, but one striking aspect is already apparent from the results presented above on the shape of the universe as viewed from 5-D. Our pictures of the Big Bang agree with the conventional opinion that it is a singularity in 4-D, but the manifold in which it is located is as smooth as a baby's bum in 5-D. There is no Big Bang in 5-D. Wow. ●

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